

HOMEWORK 8

Instruction: Please complete the first 10 problems. The last 5 problems are for you to practice, they are candidates for tests. Please print the homework and work on it directly (It is up to you whether to print it double sided or not). Due on Monday, Mar 26

Problem 1. Two cards are chosen at random from a deck of 52 playing cards. What is the probability that they

- (1) are both aces;
- (2) have the same value?

Problem 2. A woman has n keys, of which one will open her door:

- (1) If she tries the keys at random, discarding those that do not work, what is the probability that she will open the door on her k th try?
- (2) What if she does not discard previously tried keys?

Problem 3. If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day? How large need n to be so that this probability is less than 0.5?

Problem 4. What is the probability that both children of Ms. Jackson are boys provided

- (1) at least one child is a boy;
- (2) one of her children is a boy.

Problem 5. In a primitive society, every couple prefers to have a baby girl. There is a 50% chance that each child they have is a girl, and the genders of their children are mutually independent. If each couple insists on having more children until they get a girl and once they have a girl they will stop having more children, what will eventually happen to the fraction of girls in this society?

Problem 6. If you have an unfair coin, which may bias toward either heads or tails at an unknown probability. Generate even odds using this coin.

Problem 7. We throw 3 dice one by one. What is the probability that we obtain 3 points in strictly increasing order?

Problem 8. There is one amoeba in a pond. After every minute the amoeba may die, stay the same, split into two or split into three with equal probability. All its offspring, if it has any, will behave the same. What is the probability the amoeba population will die out?

Problem 9. In answering a question on a multiple-choice test, a student either knows the answer or guesses. Let p be the probability that the student knows the answer and $1-p$ the probability that the student guesses. Assume that a student who guesses at the answer will be correct with probability $1/m$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question, given that he or she answered it correctly?

Problem 10. Show that if $P(A | B) = 1$, then $P(B^c | A^c) = 1$.

Problem 11. An urn contains n white and m black balls. The balls are withdrawn one at a time until only those of the same color are left. Show that with probability $n/(n+m)$ they are all white (Hint: Imagine that the experiment continues until all the balls are removed and consider the last ball withdrawn.)

Problem 12. Let Q_n denote the probability that in n tosses of a fair coin no run of consecutive heads appears. Show that

$$Q_n = \frac{1}{2}Q_{n-1} + \frac{1}{4}Q_{n-2} + \frac{1}{8}Q_{n-3}, \quad Q_0 = Q_1 = Q_2 = 1.$$

Problem 13. (The Ballot Problem). In an election, candidate A receives n votes and candidate B receives m votes, where $n > m$. Assuming that all of the $(n + m)!/n!m!$ ordering of the votes are equally likely, let $P_{n,m}$ denote the probability that A is always ahead in the counting of the votes.

- (1) Compute $P_{2,1}, P_{3,1}, P_{3,2}$.
- (2) Find $P_{n,1}, P_{n,2}$.
- (3) Based on your results in part (1) and (2), conjecture the value of $P_{n,m}$.
- (4) Derive a recursion for $P_{n,m}$ in terms of $P_{n-1,m}$ and $P_{n,m-1}$ by conditioning on who receives the last vote.
- (5) Verify your conjecture by induction.

Problem 14. When coin A is flipped it comes up heads with probability $1/4$, whereas when coin B is flipped it comes up heads with probability $3/4$. Suppose that one of these coins is randomly chosen and is flipped twice. If both flips land heads, what is the probability that coin B was the one flipped?

Problem 15. If 4 couples are arranged in a row, find the probability that no husband sits next to his wife.