MATH 2602, Final Exam, Aug 2, 2011

Name: ______________________________

GTID: ______________________________

Section: ______________________________

Problem	*Points*
1	
2	
3	
4	
5	
6	
7	
8	

TOTAL:_________________

Please do show all your work including intermediate steps and also explain in words how you solve a problem. Partial credits are available.

Problem 1. (6pt+6pt+6pt) Consider the following LP problem

$$\begin{aligned} \min z = -x_1 - 2x_2 + 3, \\ \text{s.t.} \qquad x_1 - x_2 \geq 1 \\ x_1 \leq 2 \\ 2x_1 - 3x_2 \leq 6 \\ -x_1 - x_2 \leq 0 \\ x_1 \geq 0, x_2 \leq 0 \end{aligned}$$

(1). Convert the problem into cannonical form P_c.

(2). Show that the feasible region of P_c is convex.

(3). Solve P_c using graphical method.

Problem 2. (4pt+4pt+4pt)

(1) In how many ways can five different mathematics books, three different physics books, and four different chemistry books be arranged on a shelf?

(2) Repeat (1) if three of the five mathematics books are the same.

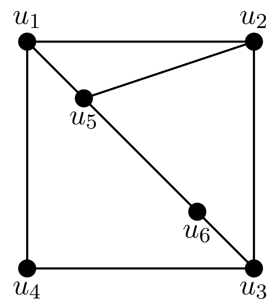
(3) How many solutions does the following equation have?

$$x_1 + 3x_2 + x_3 = 15$$

where x_1, x_2, x_3 are nonnegative integers.

Problem 3. (3pt+3pt+4pt) Given the following graph G,

(1) Find the adjacency matrix A of G;



G

(2) Find a Hamiltonian cycle in G;

(3) Find the $(5, 6)$ entry of A^4;

Problem 4. (3pt+4pt+5pt) Let G be a connected plane graph with $V \geq 3$ vertices, E edges and R regions.

(1) Count the number of edges on the boundary of each region and sum over all regions. Denote the sum by N. Show that $N \leq 2E$ and $N \geq 3R$;

(2) Show that $E \leq 3V - 6$;

(3) Show that any planar graph all of whose vertices have degree at least 5 must have at least 12 vertices. [Hint: Use (2) and Euler's theorem.]

Problem 5. (4pt+8pt)Consider the following LP problem

$$\begin{aligned} \max z &= 3x_1 + 2x_2 + 3x_3 \\ \text{s.t.} \quad 2x_1 + x_2 + x_3 &\leq 2 \\ 3x_1 + 4x_2 + 2x_3 &\geq 8 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

(1). Find the dual of this LP;

(2). Use either big-M or two-phase method to solve this LP problem.

Problem 6. (2pt+4pt+6pt)Let

$$a_n = (3+\sqrt{5})^n + (3-\sqrt{5})^n.$$

(1) Compute a_1 and a_2;

(2) Find a recursion relation that a_n satisfies;

(3) Use induction to show that $2^n \mid a_n$.

Problem 7. (3pt+3pt+2pt+6pt)We are going to use the principle of inclusion-exclusion to find the number of intergral solutions of the following equation

$$x_1 + x_2 + x_3 = 10$$

which satisfy the conditions

$$0 \le x_1 \le 4, \quad 0 \le x_2 \le 2, \quad 0 \le x_3 \le 6.$$

Let

$$\begin{aligned}
U &= \{ (x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 10, x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \}, \\
A_1 &= \{ (x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 10, x_1 \ge 5, x_2 \ge 0, x_3 \ge 0 \}, \\
A_2 &= \{ (x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 10, x_1 \ge 0, x_2 \ge 3, x_3 \ge 0 \}, \\
A_3 &= \{ (x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 10, x_1 \ge 0, x_2 \ge 0, x_3 \ge 7 \}.
\end{aligned}$$

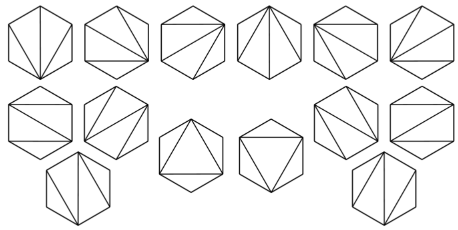
(1). Find $|U|$, $|A_1|$, $|A_2|$, $|A_3|$.

(2). Find $|A_1 \cap A_2|$, $|A_1 \cap A_3|$, $|A_2 \cap A_3|$.

(3). Find $|A_1 \cap A_2 \cap A_3|$.

(4). Solve the original problem, i.e. find $|A_1^c \cap A_2^c \cap A_3^c|$.

Problem 8. (3pt+4pt+3pt)Let C_n denote the number of different ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines. For example ,the following picture shows that $C_4 = 14$



(1). Show that

$$C_0 = 1, \quad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \quad \text{for } n \geq 0;$$

[Hint: pick out a fixed edge, this edge belongs to exactly one triangle after triangulation and this triangle splits the polygon into two parts.]

(2). Let $c(x)$ be the generating function of C_n. Show that

$$c(x) = 1 + xc(x)^2.$$

and then solve for $c(x)$;
[Hint: show that the coefficients of x^n are equal on both sides. To solve for $c(x)$, treat $c(x)$ as unknown, x as a constant, this is a quadratic equation.]

(3). Use the following Taylor expansion to expand $c(x)$ and find C_n

$$\sqrt{1+y} = 1 - 2\sum_{n=1}^{\infty} \binom{2n-2}{n-1} (-\frac{1}{4})^n \frac{y^n}{n}.$$